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Global stability of time-dependent flows. Part 2. Modulated fluid layers

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The method of energy is used to develop two stability criteria for a large class of modulated Bénard problems. Both criteria give stability limits which hold for disturbances of arbitrary amplitude. The first of these, designated as strong global stability, requires the energy of all disturbances to decay monotonically and exponentially in time. Application of this criterion results in a prediction of Rayleigh numbers below which the diffusive stagnant solution to the Boussinesq equations is unique. The second criterion requires only that disturbances decay asymptotically to zero over many cycles of modulation, and is a weaker concept of stability. Computational results using both criteria are given for a wide range of specific cases for which linear asymptotic stability results are available, and it is seen that the energy and linear limits often lie close to one another.

1. Introduction

Following its modern reformulation by Serrin (1959) and Joseph (1965, 1966), the energy stability method has found increased utility in providing sufficient conditions for stability of a large number of steady flows. In a previous paper (Homsy 1973), the method was applied to the unsteady problem of impulsive heating or cooling of fluid layers. The primary result appears as a demarcation of regions in the Rayleigh number–time plane for which one can ensure stability of the diffusive temperature field to disturbances of arbitrary amplitude. It was shown that, for Rayleigh numbers below those calculated therein, all disturbances decay exponentially in time, and hence the diffusive solution is the unique solution to the Boussinesq equations. In the present paper, the treatment is extended to the related problem of fluid layers subject to modulations in either heating, cooling or gravity.

It seems appropriate at the outset to provide a set of definitions which will facilitate the discussion below. Several notions of stability are in use by various authors. We shall refer to a flow as *strongly globally stable* if disturbances of any magnitude decay exponentially in time, in the mean. It follows that a strongly globally stable flow constitutes a unique solution to the relevant governing equations. By *asymptotically stable*, we mean those flows for which disturbances of arbitrary magnitude, while not excluded from growing, must ultimately decay in the mean as time increases. With reference to the class of modulated flows under discussion here, such a flow is *asymptotically stable* if the net growth of any disturbance, taken over one cycle of modulation, is negative. *Linear asymptotic stability*, then, will imply similar statements regarding the net growth over one cycle, but with the additional restriction that disturbances must remain small enough for the linearized equations to hold. Clearly, a flow may be regarded as 'stable' or 'unstable' according to the particular definition adopted and, as we shall see below, there are circumstances under which one usage may be preferable to others.

There exist a number of theoretical studies treating the linear asymptotic stability of modulated fluid layers, notably Venezian (1969), Rosenblat & Herbert (1970), Gresho & Sani (1970), Rosenblat & Tanaka (1971) and Yih & Li (1972). All of these papers deal with modulated Bénard problems of one type or another, and all have adopted as the stability criterion zero net growth over one cycle. As is well known, the mathematical treatment centres around the use of Galerkin's method to describe the spatial dependence of the disturbances, and the application of Floquet theory to describe the asymptotic stability characteristics of the resulting ordinary differential equations. A survey of these results indicates a predicted stabilization of the layer over ranges of frequency and amplitude of the modulation. The stabilization increases with increasing amplitude up to the point at which a subharmonic response limits the degree of stabilization. At higher amplitudes, the layer may be destabilized (Yih & Li 1972), or may approach a limiting degree of stabilization (Gresho & Sani 1970). A second general feature of these results is that the degree of stabilization is apparently a maximum in the limit of very low frequency, and becomes small as the frequency increases (Venezian 1969; Rosenblat & Tanaka 1971). This effect is a consequence of the stability criterion adopted, and leads to a pair of questions which has been recognized and discussed by the above authors: (*a*) does the stability criterion have any physical relevance if one cycle exceeds the lifetime of an observer; and (*b*) does a disturbance grow over the unstable part of the cycle to an amplitude which would invalidate the use of linear theory?

In addition to these questions, there is a related consideration which derives from the experiments of Donnelly (1964) on modulated Taylor–Couette flow. In a study designed to test for the occurrence of Taylor vortices which persisted in time, he found that the degree of stabilization observed was negligible in the limits of both zero and large frequencies.

Rosenblat & Herbert (1970) discussed and attempted to deal with these questions and in doing so proposed another definition of 'stability' in addition to the usual periodicity criterion. They introduced an 'amplitude' criterion which defines the flow as 'stable' if, during the unstable portion of the cycle, the amplitude of a disturbance does not grow beyond a prescribed value. This criterion suffers the same defects as does amplification theory as applied to impulsively started problems, namely that amplitudes are determined only *relative to an initial value*, and the 'critical' relative amplitude remains indeterminate (Homsy 1973).

It appears, then, that there are circumstances under which the usual linear theory yields stability limits which are inapplicable for a number of reasons. It is therefore of interest to develop and examine other stability criteria which complement those already available and which may serve as more attractive definitions of 'stability' in certain cases. There are two constraints present in the linear theory which might be relaxed: that of small amplitude disturbances and the periodicity condition. The restriction posed by the periodicity criterion has been removed in the low frequency or quasi-static limit by Rosenblat & Herbert (1970). They prove the intuitive result that modulated layers are stable to infinitesimal disturbances if the Rayleigh number is at all times below the classical value for steady heating. This result follows from the fact that in the quasi-static limit the temperature gradient is constant in space, with an amplitude which is time-dependent.

In the present paper, we seek the strong stability and asymptotic stability criteria using the method of energy. Energy methods have been previously applied to modulated Taylor–Couette flow by Conrad & Criminale (1965) and to the oscillatory Stokes layer by von Kerczek & Davis (1972). Conrad & Criminale developed strong stability criteria for a *restricted* class of axisymmetric disturbances under the narrow-gap assumption. These results are hence not global limits for the flow. Von Kerczek & Davis gave both strong global stability limits for three-dimensional disturbances and strong stability results for disturbances constrained to be two-dimensional. The present work reports the corresponding strong global stability limits for modulated layers. The results appear in the form of a critical Rayleigh number, dependent only upon the amplitude and frequency of the modulation, below which the diffusive stagnant solution to the Boussinesq equations is unique. An adjunct of the calculation is an extension of the Rosenblat–Herbert quasi-static result to disturbances of arbitrary amplitude.

In addition to the results mentioned above, Davis & von Kerczek (1973) have recently presented a reformulation of the energy method which allows the treatment of asymptotic stability of modulated flows, i.e. no net growth of arbitrary-sized disturbances over one cycle. For steady flows, the reformulation reduces to the previous results of Serrin (1959) and Joseph (1966). We have developed the corresponding treatment for modulated layers, which results in stability limits less stringent than the strong global limits alluded to above. In addition, the new formulation of the energy theory results in the stability boundary now becoming a function of the Prandtl number as well as the amplitude and frequency of modulation. These new results, taken together with those previously available, give a fairly complete description of the stability limits for this class of problems for a variety of stability criteria and specific modulation histories. Of particular interest is the fact that the energy and linear stability limits are, under some circumstances, close, thus appreciably restricting the regions in parameter space for which subcritical convection is possible.

In §2 we define the basic state whose stability is to be examined, and present the energy identities. Section 3 consists of a formalization of the definitions of stability which we advanced above. The ideas are illustrated for the case of gravity modulation in §4, since the results there are particularly straightforward. In §5 we present detailed computational results for two choices of surface temperature modulations, and we conclude with a discussion of the results and a comparison with available experiments in §6.

2. Preliminaries

The base state

In order to be able to make a comparison with the previously developed linear-theory results, we shall treat a wide class of modulated Bénard problems, and we shall make the appropriate reductions to special cases in later sections. Consider an infinite fluid layer bounded by surfaces at $z' = 0, d$ respectively, and let the surface temperatures be given in dimensional form by

$$T = \begin{cases} T_0 + T_s \cos(\omega' t'), & z' = 0, \\ T_1 + \delta T_s \cos(\omega' t'), & z' = d. \end{cases}$$

Furthermore, let the layer be subject to gravity modulation of amplitude ϵg. Then let the variables be scaled according to $\{z, t, \theta\} = \{d, d^2/\kappa, T_0 - T_1\}$. The Boussinesq equations then admit a stagnant solution in which the dimensionless temperature $\bar{\theta} = (T - T_1)/(T_0 - T_1)$ satisfies the diffusion equation

$$\partial\bar{\theta}/\partial t = \partial^2\bar{\theta}/\partial z^2, \tag{2.1}$$

subject to the conditions

$$\theta = \left\{ \begin{matrix} 1 + a\cos(\omega t), & z = 0, \\ \delta a \cos(\omega t), & z = 1. \end{matrix} \right\} \tag{2.2}$$

Here $a = T_s/(T_0 - T_1)$ and will be referred to as the modulation amplitude, and $\omega = \omega'\kappa/d^2$ is a dimensionless modulation frequency. In addition to (2.1) and (2.2), the base state consists of a hydrostatic pressure field whose exact form is of no consequence in what follows.

The solution to (2.1) and (2.2) is easily found to be

$$\bar{\theta} = 1 - z + a\,\mathrm{Re}\left\{ e^{i\omega t} \frac{\delta \sinh(\beta z) + \sinh[\beta(1-z)]}{\sinh\beta} \right\}, \tag{2.3}$$

$$\beta = (i\omega)^{\frac{1}{2}} = (1+i)(\tfrac{1}{2}\omega)^{\frac{1}{2}}.$$

Thus $\delta = -1$ ($\epsilon = 0$) corresponds to the case considered by Yih & Li, and $\delta = 0$ ($\epsilon = 0$) to that of Rosenblat & Tanaka. The effect of gravity modulation will appear in the dynamic equations themselves. It is this base state for which we shall develop stability limits below.

The energy identities

For the disturbances, let the scalings be

$$\{\mathbf{r}, t, \mathbf{v}, p, \theta\} = \{d, d^2/\kappa, \kappa/d, \rho\kappa\nu/d^2, \Delta T\}.$$

The nonlinear Boussinesq equations then take the dimensionless form

$$\sigma^{-1}(\partial\mathbf{v}/\partial t + \mathbf{v}\,.\,\nabla\mathbf{v}) = -\nabla p + \nabla^2\mathbf{v} + Ra\theta\mathbf{k}(1 + \epsilon\sin\omega t), \tag{2.4}$$

$$\partial\theta/\partial t + \mathbf{v}\,.\,\nabla\theta = \nabla^2\theta - w\partial\bar{\theta}/\partial z. \tag{2.5}$$

The notation is standard; σ is the Prandtl number, Ra the Rayleigh number, $\bar{\theta}(z, t)$ the base state (2.2), ω the dimensionless frequency, ϵ the amplitude of the gravity modulation, and $w = \mathbf{k}\,.\,\mathbf{v}$ the vertical velocity. The case treated by

Gresho & Sani is thus given by $a = 0$, $\epsilon \neq 0$. For planar boundaries across which there is no flow, and which, for convenience, we take to be conducting, one can derive the *energy evolution equation* in its symmetric form (cf. Homsy 1973)

$$\frac{dE}{dt} \equiv \frac{d}{dt}\left(\frac{\langle|\mathbf{v}|^2\rangle}{2\sigma}+\frac{\langle\phi^2\rangle}{2}\right)$$
$$= Ra^{\frac{1}{2}}\left\{\frac{\langle w\phi(1+\epsilon\sin\omega t)\rangle}{\lambda^{\frac{1}{2}}}-\lambda^{\frac{1}{2}}\left\langle w\phi\frac{\partial\bar{\theta}}{\partial z}\right\rangle\right\}-\langle\nabla\mathbf{v}:\nabla\mathbf{v}+|\nabla\phi|^2\rangle \quad (\lambda \geqslant 0). \quad (2.6)$$

The angular brackets denote volume integration over the layer. We may also write (2.6) as

$$dE/dt = Ra^{\frac{1}{2}}\, I_\lambda(t) - D, \quad (2.7)$$

where I_λ and D are defined accordingly. Equation (2.6) is exact and holds for any solution to the Boussinesq equations. It is from (2.3) and (2.6) that the stability criteria will be developed.

3. Stability criteria

In this section we express mathematically the two stability criteria which form the basis of this paper, viz. strong global stability and asymptotic stability. In the process of doing so, we also discuss the optimal stability boundary, which gives the method by which the free positive constant λ is chosen.

Strong global stability

Strong global stability, and hence uniqueness of the base state (2.3), is proved in the usual manner (cf. von Kerczek & Davis 1972). From (2.6) we first deduce the energy inequality

$$D^{-1}\, dE/dt \leqslant -1 + Ra^{\frac{1}{2}}/\rho_\lambda. \quad (3.1)$$

Inequality (3.1) holds whenever ρ_λ is a solution to the maximum problem

$$\frac{1}{\rho_\lambda} = \max_{h}\left\{\frac{\left\langle w\phi\left(\frac{1+\epsilon\sin\omega t}{\lambda^{\frac{1}{2}}}-\lambda^{\frac{1}{2}}\frac{\partial\bar{\theta}}{\partial z}\right)\right\rangle}{D}\right\}, \quad (3.2)$$

$h = \{\mathbf{v},\phi\,|\,\nabla\cdot\mathbf{v} = 0;\ (\mathbf{v},\phi)\in C^2$, Fourier-transformable or periodic in the horizontal plane; $\mathbf{v} = \phi = 0$ at $z = 0, 1\}$.

Clearly ρ_λ is parametrically a periodic function of time with period $2\pi/\omega$. Denote the minimum of this function by $R_{S,\lambda}$, i.e.

$$R_{S,\lambda} = \min_{t\in[0,\,2\pi/\omega]} \rho_\lambda(t). \quad (3.3)$$

Then it is not difficult to prove that the layer is strongly stable for Rayleigh numbers such that $Ra^{\frac{1}{2}} < R_{S,\lambda}$. From (3.1) we have

$$D^{-1}\, dE/dt \leqslant -1 + Ra^{\frac{1}{2}}/R_{S,\lambda}. \quad (3.4)$$

Couple this with the inequality $D \geqslant \xi^2 E$ (Joseph 1965) to find

$$E^{-1} dE/dt \leqslant \xi^2(-1 + Ra^{\frac{1}{2}}/R_{S,\lambda}), \tag{3.5}$$

or

$$E(t) = E(0)\exp\{-\xi^2(1 - Ra^{\frac{1}{2}}/R_{S,\lambda})\,t\}. \tag{3.6}$$

Thus for $Ra^{\frac{1}{2}} \leqslant R_{S,\lambda}$ the energy of any disturbance *always* decays exponentially in time. The optimal stability limit is clearly given by choosing λ so that $R_{S,\lambda}$ is as large as possible. This value of $R_{S,\lambda}$ will be denoted by R_S;

$$R_S = \max_{\lambda} \min_{t \in [0,\, 2\pi/\omega]} \rho_\lambda(t). \tag{3.7}$$

Asymptotic stability

It is clear that the strong stability results computed in the above manner are the best possible if one requires that disturbances should *always decrease*. Davis & von Kerczek (1973) have shown, however, that one can develop a criterion which ensures that although disturbances may grow to a large amplitude over part of one cycle, their magnitude decreases asymptotically to zero over many cycles. Their ideas may be readily adapted for the present discussion. Interestingly enough, if one adopts asymptotic stability as the criterion, the Prandtl number reappears as a parameter of the problem. The essence of the approach is to avoid the use of the isoperimetric inequality $D \geqslant \xi^2 E$, and thus obviate the need to carry out the minimization in (3.3).

Return to the energy evolution equation (2.7):

$$dE/dt = Ra^{\frac{1}{2}} I_\lambda(t) - D.$$

Now consider the maximum problem

$$\nu_\lambda(t) = \max_{h} \left\{\frac{(Ra^{\frac{1}{2}} I_\lambda(t) - D)}{E}\right\} \tag{3.8}$$

for any fixed Rayleigh number Ra. Then combining (2.6) and (3.8) we deduce the energy inequality

$$dE/dt \leqslant \nu_\lambda(t)\, E. \tag{3.9}$$

It is essential to note that, since we have not required the introduction of the isoperimetric inequality, (3.9) is a bound for the energy *growth* as well as the decay of all kinematically admissible functions in h. Clearly $\nu_\lambda(t)$ is again a periodic function of t. Integration of (3.9) yields

$$E(t) \leqslant E(0)\exp\left[\int_0^t \nu_\lambda(\tau)\, d\tau\right]. \tag{3.10}$$

Accordingly, the base state is asymptotically stable if

$$\int_0^{2\pi/\omega} \nu_\lambda(\tau)\, d\tau < 0, \tag{3.11a}$$

or equivalently

$$E(t + 2\pi/\omega)/E(t) < 1{\cdot}0. \tag{3.11b}$$

Now the optimum stability boundary is determined by choosing λ so that $E(2\pi/\omega)/E(0)$ is as small as possible, and the energy limit for this criterion is given by the largest Rayleigh number Ra for which (3.11) holds. We shall denote this stability limit by R_A, and the layer is hence asymptotically stable in the mean for $Ra^{\frac{1}{2}} \leqslant R_A$.

4. Gravity modulation

We have chosen to treat the case of gravity modulation separately; some general results may be obtained without detailed computation and we feel that displaying these may help to bring out various features of the criteria advanced above. We thus set $a = 0$ above, and the energy evolution equation reduces to

$$\frac{dE}{dt} = Ra^{\frac{1}{2}} \langle w\phi \rangle \left(\frac{1+\epsilon \sin \omega t + \lambda}{\lambda^{\frac{1}{2}}} \right) - D \tag{4.1}$$

since

$$\partial \bar{\theta}/\partial z = -1.$$

Consider first the strong stability limit, for which one considers the following problem:

$$\frac{1}{\rho_\lambda} = \max_{h} \frac{\langle w\phi \rangle}{D} \left(\frac{1+\epsilon \sin \omega t + \lambda}{\lambda^{\frac{1}{2}}} \right). \tag{4.2}$$

The Euler–Lagrange equations for the maximum problem (4.2) are found to be

$$\nabla . \mathbf{v} = 0, \tag{4.3a}$$

$$\nabla^2 \mathbf{v} + \frac{\rho_\lambda}{2} \left\{ \frac{1+\epsilon \sin \omega t + \lambda}{\lambda^{\frac{1}{2}}} \right\} \phi \mathbf{k} - \nabla \pi = 0, \tag{4.3b}$$

$$\nabla^2 \phi + \frac{\rho_\lambda}{2} \left\{ \frac{1+\epsilon \sin \omega t + \lambda}{\lambda^{\frac{1}{2}}} \right\} w = 0. \tag{4.3c}$$

These equations are identical to the linear Bénard equations for the steady case with $Ra^{\frac{1}{2}}$ replaced by the factor $\frac{1}{2}\rho_\lambda \{ \}$. Thus for infinite plane layers, the solution of the Euler–Lagrange equations (4.3 a–c) for the most dangerous Fourier mode is

$$\frac{\rho_\lambda}{2} \left\{ \frac{1+\epsilon \sin \omega t + \lambda}{\lambda^{\frac{1}{2}}} \right\} = R_L, \tag{4.4}$$

where, for example, $R_L^2 = 1708$ for rigid conducting boundaries. The global limits follow from (4.4) as

$$R_S = \max_{\lambda} \min_{t} \rho_\lambda(t) = \max_{\lambda} \frac{2R_L}{(1+\epsilon+\lambda)/\lambda^{\frac{1}{2}}} = \frac{R_L}{(1+\epsilon)^{\frac{1}{2}}}. \tag{4.5}$$

Thus for Rayleigh numbers

$$Ra \leqslant R_L^2/(1+\epsilon) \tag{4.6}$$

the layer is strongly globally stable. Inequality (4.6) of course has a compelling physical interpretation. It states that the gravity-modulated layer is globally stable for Rayleigh numbers below which the available potential energy due to the stratification never exceeds that necessary to initiate convection in the steady case.

We now indicate how this result may be improved upon if one relaxes the criterion to that of asymptotic stability. In this case, (3.8) becomes the maximum problem and the Euler–Lagrange equations for gravity modulation become

$$\nabla . \mathbf{v} = 0, \tag{4.7a}$$

$$\nu_\lambda \mathbf{v}/\sigma = -\nabla\pi + \nabla^2\mathbf{v} + Ra^{\frac{1}{2}}\, g(\lambda, t)\, \phi\mathbf{k}, \tag{4.7b}$$

$$\nu_\lambda \phi = \nabla^2\phi + Ra^2\, g(\lambda, t)\, w, \tag{4.7c}$$

where $g(\lambda, t) = (1 + \epsilon \sin \omega t + \lambda)/2\lambda^{\frac{1}{2}}$. We have not been able to solve (4.7) in general. Below we give a calculation for 'free–free' boundaries, in which case the solution may be developed in some detail. It is expected that the results for other boundary conditions would not differ in essential features.† Equations (4.7*a*–*c*) may be combined in the usual way to yield a system in (w, ϕ) alone. To (4.7*c*) we append

$$\nu_\lambda \nabla^2 w/\sigma = \nabla^4 w + Ra^{\frac{1}{2}}\, g(\lambda, t)\, \nabla_1^2 \phi, \tag{4.7d}$$

where ∇_1^2 is the horizontal Laplacian. Since (4.7*c*, *d*) are cyclic in the horizontal co-ordinates, we have for free–free boundaries

$$\left.\begin{aligned} w = \hat{w} f(x, y) \sin(\pi z), \quad \phi = \hat{\phi} f(x, y) \sin(\pi z),\\ \nabla_1^2 f = -\alpha^2 f. \end{aligned}\right\} \tag{4.8}$$

When (4.8) is substituted into (4.7*c*, *d*) and non-trivial solutions for $\hat{w}$ and $\hat{\phi}$ are required, the following relation between ν_λ, σ and Ra results:

$$\nu_\lambda^2 h/\sigma + \nu_\lambda(1 + \sigma^{-1})\, h^2 + h^3 - Ra\, \alpha^2 g^2(\lambda, t) = 0, \tag{4.9a}$$

or

$$\nu_\lambda = -\tfrac{1}{2}\{(1 + \sigma)\, h + [(1 + \sigma)^2 h^2 + 4(Ra\, \alpha^2 g^2 \sigma/h - \sigma h^2)]^{\frac{1}{2}}\}, \tag{4.9b}$$

where $h = \pi^2 + \alpha^2$ and $R_L^2 = h^3/\alpha^2 = (\pi^2 + \alpha^2)^3/\alpha^2$ is the critical Rayleigh number for these boundary conditions for *steady* heating from below (Chandrasekhar 1961, p. 35). Some algebraic manipulation yields

$$\nu_\lambda = \tfrac{1}{2} h\{-(\sigma + 1) + [(1 - \sigma)^2 + 4\sigma(Ra/R_L^2)\, g^2(\lambda, t)]^{\frac{1}{2}}\}. \tag{4.10}$$

It has not been possible to proceed at this level of generality and apply the periodicity condition (3.11*a*): it is possible, however, to treat the special cases $\sigma = 1$, $\sigma \to 0$ and $\sigma \to \infty$. For $\sigma = 1$, we have

$$\frac{\nu_\lambda}{h} = -1 + \frac{Ra^{\frac{1}{2}}}{R_L}\left(\frac{1 + \epsilon \sin \omega t + \lambda}{2\lambda^{\frac{1}{2}}}\right). \tag{4.11}$$

Now the criterion for asymptotic stability is

$$\int_0^{2\pi/\omega} \nu_\lambda(\tau)\, d\tau \leqslant 0,$$

which is satisfied whenever

$$Ra^{\frac{1}{2}}/R_L < 2\lambda^{\frac{1}{2}}/(1 + \lambda). \tag{4.12}$$

The value of λ which makes the region of stability as large as possible is simply $\lambda = 1$; the layer is asymptotically stable for Rayleigh numbers less than the linear limit for steady heating, i.e. $R_A = R_L$ at $\sigma = 1$.

† I am indebted to Prof. Stephen H. Davis for providing the details of this calculation.

Consider now the case $\sigma \to \infty$; (4.9 *a*) yields

$$\nu_\lambda = (\alpha^2/h^2)\,(Ra\,g^2(\lambda,t) - R_L^2) \tag{4.13}$$

and the periodicity criterion yields

$$Ra \leqslant 4\lambda R_L^2/[(1+\lambda)^2 + \tfrac{1}{2}\epsilon^2]. \tag{4.14}$$

Taking λ as the value which maximizes the region of stability ($\lambda^2 = 1+\frac{1}{2}\epsilon^2$), we have the optimum asymptotic limit

$$R_A = 2R_L/[1+(1+\tfrac{1}{2}\epsilon^2)^{\frac{1}{2}}] \quad \text{as} \quad \sigma \to \infty. \tag{4.15}$$

The case $\sigma \to 0$ proceeds in the same manner. From (4.9 *a*)

$$\nu_\lambda = \sigma(\alpha^2/h^2)\,(Ra\,g^2(\lambda,t) - R_L^2), \tag{4.16}$$

which is identical to (4.13) except for a multiplicative factor of σ. This factor cannot alter the stability limit given by the periodicity condition, and thus we have, as before,

$$R_A = 2R_L/[1+(1+\tfrac{1}{2}\epsilon^2)^{\frac{1}{2}}] \quad \text{as} \quad \sigma \to 0. \tag{4.17}$$

We note here for future reference that (4.9 *a*) is symmetric in the following sense: if (ν_λ, σ) is a solution to (4.9 *a*) then $(\nu_\lambda\sigma, \sigma^{-1})$ is also a solution. Since we subsequently invoke the periodicity criterion, the factor of σ in the latter solution does not affect the marginal curve on which $\int \nu_\lambda d\tau = 0$. Thus we conclude that $R_A(\sigma) = R_A(\sigma^{-1})$. We also note that these results complement those of Gresho & Sani (1970) by giving the lower limits for which gravity modulation may destabilize a fluid layer. Gresho & Sani suggest that such a destabilization might occur, but they provide no calculations which support this conjecture. Finally, the stability limits are such that $R_S \leqslant R_A \leqslant R_L$, as must be the case.

5. Surface temperature modulation

We now present the results of detailed computations for cases when the surface temperature is modulated, i.e. $a \neq 0$, $\epsilon = 0$. The computations have been restricted to the cases for which linear-theory results have been obtained, viz. rigid–rigid boundaries with $\delta = 0$ (Rosenblat & Tanaka 1971) and $\delta = -1$ (Yih & Li 1972).

The quasi-static limit

There is a direct analogy between the cases of gravity and surface temperature modulation in the limit of very low frequency ($\omega \to 0$) which we wish to exploit here before proceeding to the general case. First consider strong global stability. The Euler–Lagrange equations corresponding to (3.2) for $\epsilon = 0$ become

$$\nabla \cdot \mathbf{v} = 0, \tag{5.1a}$$

$$\nabla^2\mathbf{v} + \frac{\rho_\lambda}{2}\left\{\frac{1+\lambda+a\lambda g(z,t)}{\lambda^{\frac{1}{2}}}\right\}\phi\mathbf{k} - \nabla\pi = 0, \tag{5.1b}$$

$$\nabla^2\phi + \frac{\rho_\lambda}{2}\left\{\frac{1+\lambda+a\lambda g(z,t)}{\lambda^{\frac{1}{2}}}\right\}w = 0, \tag{5.1c}$$

where we have written

$$\partial\bar{\theta}/\partial z = -1 - a\mathrm{Re}\,\{e^{i\omega t}\beta[\cosh(\beta(1-z)) - \delta\cosh(\beta z)]/\sinh\beta\}$$
$$= -1 - ag(z,t). \tag{5.2}$$

Now for small ω, an expansion about $\omega = \beta = 0$ gives

$$g(z,t) = \mathrm{Re}\,[e^{i\omega t}\{(1-\delta) + i\omega(\tfrac{1}{2}(1-z^2) - \tfrac{1}{6}) + \ldots\}] \tag{5.3}$$

and thus in the limit $\omega = 0$ the base-state temperature gradient is quasi-static, i.e. it is independent of z with a modulated amplitude

$$\partial\bar{\theta}/\partial z = -1 - a(1-\delta)\cos\omega t + O(\omega). \tag{5.4}$$

In this case, equations (5.1) again reduce to the linear Bénard equations for the steady case, and one finds, in analogy with the discussion following (4.2), that

$$R_S = R_L/[1 + a(1-\delta)]^{\frac{1}{2}}. \tag{5.5}$$

The Rosenblat–Herbert quasi-static result is thus extended to all finite amplitude disturbances. We have carried out a perturbation expansion in ω, but the details are not worth pursuing here, since they are pre-empted by the numerical calculations presented below.

It is also possible to show that, in the quasi-static limit, the asymptotic stability criteria for gravity and surface temperature modulation coincide. Of course, the practicality of the asymptotic criterion can be called into question in this case, since the period of one cycle may be longer than the lifetime of an observer. It is well to record the results here anyway, since they serve as limiting cases. For *free–free* boundaries in the quasi-static limit, it is possible to show that

$$R_A = \begin{cases} 2R_L/[1 + (1 + \frac{1}{2}\epsilon^2(1-\delta)^2)^{\frac{1}{2}}], & \sigma \to 0, \infty, \\ R_L, \quad \sigma = 1\cdot 0, \end{cases}$$

and

$$R_A(\sigma) = R_A(-\sigma).$$

Strong global stability

A fairly complete parametric study was done for the strong stability results for both $\delta = 0$ and $\delta = -1$. The system (5.1 a, b) may be combined to give, in addition to (5.1 c),

$$\frac{\rho_\lambda}{2}\left(\frac{1 + \lambda + a\lambda g(z,t)}{\lambda^{\frac{1}{2}}}\right)\nabla_1^2\phi + \nabla^4 w = 0. \tag{5.1d}$$

Now (5.1 c, d) are cyclic in the horizontal co-ordinates, so w and ϕ can be Fourier transformed into modes characterized by a single wavenumber α. The z-dependent parts of the transforms $\hat{w}$ and $\hat{\phi}$ were then expanded in trial functions which consisted of sines for $\hat{\phi}$ and 'beam' functions for $\hat{w}$ (see Chandrasekhar 1961, appendix 5). As in Homsy (1973), the expansion resulted in an algebraic eigenvalue problem for $\rho_\lambda\,(t;\alpha)$. If N terms are taken in each expansion the resulting $2N \times 2N$ problem may be reduced to an $N \times N$ problem as before. Up to ten terms were taken in each expansion, and the optimum stability boundary was computed as

$$R_s = \min_t \min_\alpha \max_\lambda \rho_\lambda(t;\alpha),$$

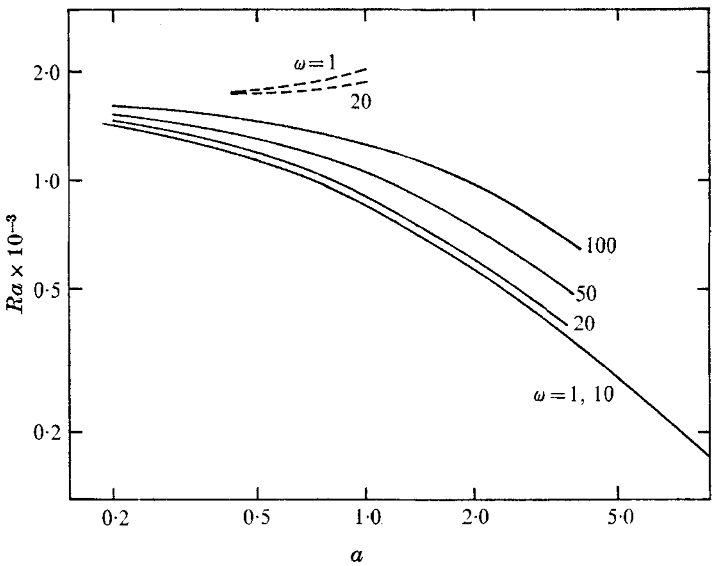

FIGURE 1. The stability limits for modulation of the bottom surface temperature. ——, strong global stability limits; – – – –, linear asymptotic stability limits of Rosenblat & Taneka (1971) for $\sigma = 1{\cdot}0$.

using standard optimization routines. Details of the computational procedure are available from the author upon request.

The results for bottom-plate modulation ($\delta = 0$) are shown in figure 1. For $\omega = 0, 1$ and 10 the computed results are very close to the quasi-static limit $R_S^2 = R_L^2/(1+a)$, i.e. $R_S^2 = 1708/(1+a)$. For higher frequencies the stability limits increase, which is a consequence of the fact that, as ω becomes large, the function $g(z,t)$ describing the time-dependent part of the gradient develops a boundary-layer character. At very large frequencies in fact, $g(z,t)$ is zero everywhere except in a layer of dimensionless thickness $O(\omega^{\frac{1}{2}})$ near the bottom boundary. Thus it is reasonable to expect that $R_S \to R_L$ as $\omega \to \infty$; the trend of the computed curves bears this out. The behaviour at high amplitude is also consistent with previous work. Consider the case of high amplitude and high frequency. Let the time t be fixed at that value t^* for which the minimum of ρ_λ occurs. Then the problem (5.1) *for that fixed time* is mathematically (but not physically) analogous to the energy stability theory for a layer with non-uniform heat sources: the strength of the source is given by a and the spatial structure by the function $g(z,t^*)$. Joseph & Shir (1966) have shown that for uniform heat sources the energy limits behave as $R_E \to \text{constant}/a$ for large a. It is possible to repeat their arguments for non-uniform heat sources, but we shall not do so here. It is clear from figure 1, however, that the same relation applies to the present problem as the amplitude of the modulation becomes large. Also shown on figure 1 are the limited results of Rosenblat & Tanaka (1971) for a Prandtl number of unity, for two frequencies. Since their results indicate a maximum stabilization for $\sigma \sim 1{\cdot}0$, at least over the parameter range considered, the curves provide an upper bound for the degree of stabilization expected for $a \leqslant 1{\cdot}0$ and

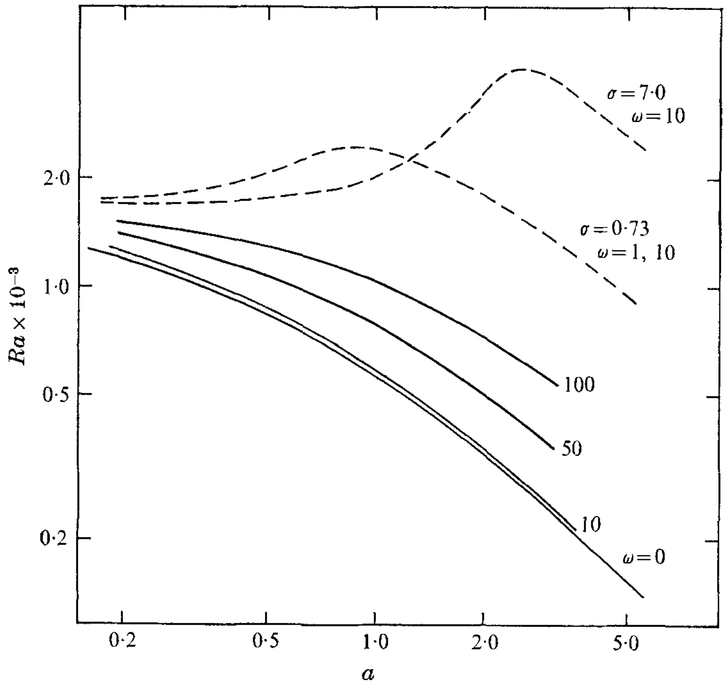

FIGURE 2. The stability limits for modulation of top and bottom temperatures. ——, strong global stability limits; – – – –, approximate envelope of linear asymptotic limits of Yih & Li (1972).

moderate frequencies. It is important to note in this and later comparisons that the stability criteria in the two cases are radically different. The linear theory gives sufficient conditions for the net growth and hence instability of small disturbances, while the strong energy limits give sufficient conditions for exponential decay of arbitrary disturbances. A more suitable comparison follows the discussion of the asymptotic stability results. Subcritical instabilities are possible only in the band between the linear and energy limits and the width of the band decreases with increasing frequency. Everywhere below the strong energy limits, the diffusive solution $\overline{\theta}(z,t)$ is unique.

The results for modulation of top and bottom surface temperatures ($\delta = -1$) are shown in figure 2. The strong energy limits are qualitatively similar to those in figure 1. Also shown in figure 2 is the approximate locus of the linear asymptotic results of Yih & Li (1972) for available values of σ and ω. The detailed curves have discontinuities at points for which the response changes from synchronous to subharmonic, but the fine details are not of interest here. It is notable that Yih & Li predict a destabilization at high modulation amplitudes which are not excluded by the energy limits. The region of allowable subcritical instabilities is larger in this case, but the stability limits still remain within an order of magnitude of each other.

Asymptotic stability

The computation of the asymptotic stability limits was more involved. For modulated surface temperatures, the Euler–Lagrange equations for the maximum problem (3.8) can be combined to yield the following set of equations for (w, ϕ):

$$\nu_\lambda \frac{\nabla^2 w}{\sigma} = \nabla^4 w + \frac{R^{\frac{1}{2}}}{2}\left(\frac{1+\lambda+a\lambda g(z,t)}{\lambda^{\frac{1}{2}}}\right)\nabla_1^2\phi, \qquad (5.6a)$$

$$\nu_\lambda \phi = \nabla^2\phi + \frac{R^{\frac{1}{2}}}{2}\left(\frac{1+\lambda+a\lambda g(z,t)}{\lambda^{\frac{1}{2}}}\right) w, \qquad (5.6b)$$

where the form of $g(z,t)$ is implied by (5.2).

To solve for $\nu_\lambda(t)$, we again applied Fourier decomposition followed by Rayleigh–Ritz expansions for $\hat{w}$ and $\hat{\phi}$. Although the resulting algebraic problem could be made symmetric, it was not possible to reduce the problem to an $N \times N$ problem as before. Thus to generate $\nu_\lambda(t)$ at a given value of t it was necessary to solve a $2N \times 2N$ generalized eigenvalue problem, which substantially increased the computation time. The computation proceeded as follows: given a set of parameters $\{\sigma, Ra, \lambda, a, \omega, \alpha\}$, the quantity

$$\bar{\nu}_\lambda = \int_{-\pi/\omega}^{\pi/\omega} \nu_\lambda(\tau)\, d\tau$$

was computed. To minimize the number of function evaluations necessary, the integral was evaluated numerically using five- or seven-point Gauss–Legendre quadrature. The next step was to compute

$$\tilde{\nu} = \min_{\lambda} \max_{\alpha} \bar{\nu}_\lambda$$

using standard optimization techniques. $\tilde{\nu}$ of course gives the amplification over one cycle of the most dangerous Fourier mode, with λ adjusted to minimize that amplification. Finally, for a given σ, a and ω, the Rayleigh number Ra was varied to find the point at which $\tilde{\nu} = 0$; the value was designated as R_A.

Three parametric studies were made by varying each of σ, a and ω with the other two held fixed. We chose to treat the Rosenblat–Tanaka boundary conditions ($\delta = 0$) as representative of the results obtained with this criterion. The first of these studies is shown in figure 3, where we have compared the linear asymptotic, asymptotic and strong stability limits for the case $\sigma = 1{\cdot}0$, $\omega = 1{\cdot}0$, with varying a. The computations of Rosenblat & Tanaka are not as extensive as they might be, but comparison of the two energy limits gives a measure of the improvement in stability limits to be obtained as one relaxes the criterion. Figure 4 shows a similar parametric study for $\sigma = a = 1{\cdot}0$, with ω varying. The linear asymptotic results exhibit a maximum in predicted stabilization as $\omega \to 0$, and as we have noted in the introduction, adoption of this criterion may not be advantageous in this limit. The strong energy limits are of course independent of Prandtl number, and rise steadily from the quasi-static limit (5.5) to the apparent asymptote $R_S = R_L$ as the effects of the modulation become

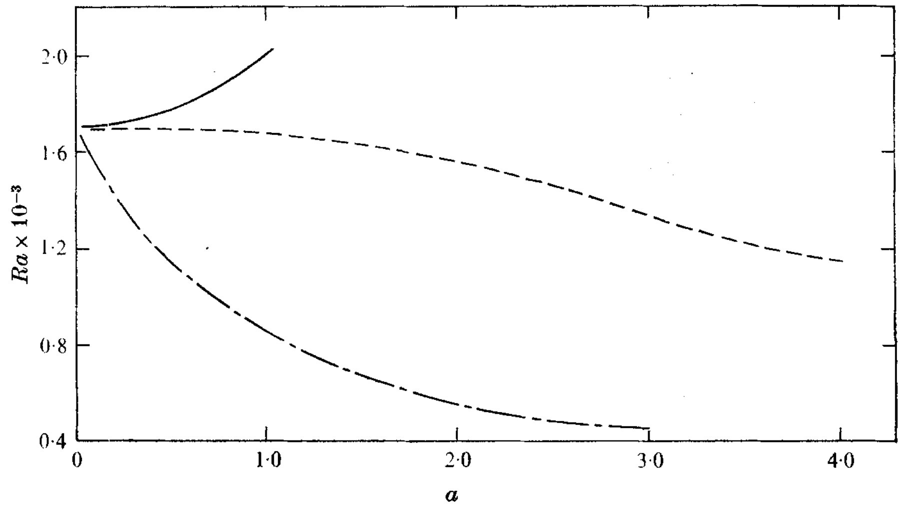

FIGURE 3. The three stability limits for modulated bottom temperatures as a function of modulation amplitude for $\sigma = \omega = 1{\cdot}0$. ——, Rosenblat & Takaka (1971); – – – –, asymptotic stability limit; — – —, strong global stability limit. The lower curve is independent of Prandtl number.

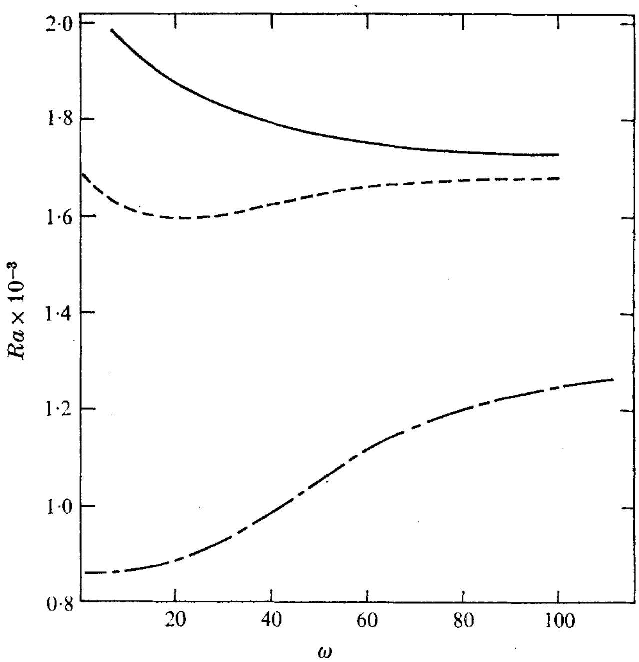

FIGURE 4. The three stability limits as a function of frequency for $\sigma = a = 1{\cdot}0$. ——, Rosenblat & Tanaka (1971); - - - - - -, asymptotic stability limit; — – — –, strong global stability limit.

σ	0·5	1·5	5·0	10·0
Ra_c (Rosenblat & Tanaka)	2090	2080	1905	—
R_A^2	1710	1670	1623	1605

TABLE 1. Comparison of linear- and energy-theory limits for asymptotic stability as a function of Prandtl number for $a = \omega = 1{\cdot}0$. The strong stability limit is $R_S^2 = 854$.

confined to a small region near the bottom plate. The asymptotic energy limits are instructive for two reasons. First, they confirm the quasi-static limit $R_A = R_L$, which we have proved only for free–free conditions. More important, they provide a more logical basis for comparison with the linear limits, since the criteria for stability are similar. It is of particular interest, then, to note that the two limits are close, and thus subcritical convection, if it occurs, is confined to Rayleigh numbers in the region between the two limits. Thus, the present class of problems appears to be one in which the linear and energy stability results strongly complement each other.

Finally, in table 1, we have compared the Prandtl number dependence for the energy and linear theories respectively. Surprisingly, the dependence appears to be similar, with both theories predicting maxima in the range $0{\cdot}5 \leqslant \sigma \leqslant 1{\cdot}0$.

6. Discussion

We have defined and developed criteria for stability of modulated Bénard problems which seek to provide alternative definitions of stability to those previously advanced. The question of the applicability of these various criteria to experimental situations naturally arises. Unfortunately, there are few experiments available for comparison. We have already mentioned the experiments of Donnelly (1964) on the related problem of modulated Taylor–Couette flow. (We have not done so here, but it is possible to show that, under the assumptions of axisymmetric disturbances and the narrow-gap approximation, the modulated Taylor–Couette problem is mathematically analogous to the theory developed above; cf. Conrad & Criminale (1965).) Previous authors, in an effort to compare Donnelly's experiments with the linear asymptotic theory, have apparently overlooked the following definition which Donnelly himself advanced for stability. "The criterion for instability with modulation was taken to be the presence of *regular* cells.... Under certain circumstances, one can find a trace of cell motion as soon as the criterion $\bar{\Omega} + \Delta\Omega = \Omega_C[Ra(1+a) = R_L^2]$ is exceeded. However,..., the [averaged] signal amplitude does not increase appreciably showing *that these are transient vortices and do not amplify*.... We take as our criterion for the onset of stability that speed at which the signal amplitude starts to increase with increasing $\bar{\Omega}[Ra]$." (The emphasis is added.) Thus Donnelly considers a flow 'stable' until the appearance of fully developed Taylor vortices of sufficient amplitude to be detected by his particular measurement technique. There exist no theoretical studies to our knowledge which use a similar criterion for stability, for to do so would be to undertake the prediction of the final equilibrated finite amplitude motion. It is also interesting to note that Donnelly considers flows which exhibit measurable motions over only part of a cycle as being 'stable', while such motions may have profound effects on the transport of heat, mass or momentum in the system. We are of the opinion, therefore, that these particular data form a poor basis for comparison with theory.

Recently, Finucane (1972) has conducted an experimental study of the onset of instability in fluid layers with modulated bottom temperature. The study was confined to low frequencies ($0 \leqslant \omega \leqslant 4{\cdot}0$) and moderate amplitudes ($0 \leqslant a \leqslant 1{\cdot}0$). 'Instability' was defined as the occurrence of any measurable *local* departure from the diffusive solution, which thus constitutes a stringent criterion indeed. For large amplitudes, the data fully support the quasi-static criterion

$$Ra \leqslant R_L^2/(1+a)$$

for stability so defined. For moderate amplitudes, $a \sim 0{\cdot}5$, $\omega < 1{\cdot}0$, the quasi-static result is again in complete agreement with the measured 'stability' curve. For slightly higher ω ($1{\cdot}0 \leqslant \omega \leqslant 3{\cdot}5$), Finucane finds some degree of stabilization, but measurable local convection still occurs for $Ra < R_L$. This is consistent with the strong energy limits given in figure 2 in the sense that convection does not occur below the energy limit. Over this small range of frequency the strong stability limits remain unchanged, while the experiments show some stabilization. This is perhaps not surprising, since care was taken in the experiments to eliminate large extraneous disturbances. In short, the experiments are consistent with the strong stability limits and most important, demonstrate the inapplicability of the periodicity criterion at low frequencies.

What is needed at present is experiments at higher ω which would indicate at what frequencies the periodicity condition becomes advantageous from a practical viewpoint.

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